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Comment on Final State Interactions beyond the Plane Wave Impulse Approximation description of inclusive scattering

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Abstract

It is shown that no set of reasonable approximations leads to a structure function as a convolution of the PWIA and a FSI contribution.

Benhar *et al* have proposed for the structure function of a composite system a form based on the Plane Wave Impulse Approximation (PWIA), corrected for Final State Interaction (FSI) [1]. Although the result has been critically reviewed before [2] a recent preprint [3] takes much the same stand and we renew an attempt to clarify the issue.

Consider the dominant incoherent part of the structure function per particle due to a density fluctuation $\rho_q(\boldsymbol{r}_1) = \exp(i\boldsymbol{q}\boldsymbol{r}_1)$ of a selected particle $'1'$

$$S(q,\omega) = (m/q)\phi(q,y) = -(1/\pi)\mathrm{Im}\Big\langle \Phi_A^0|\rho_q^\dagger(\boldsymbol{r}_1)G(\omega + E_A^0 + i\epsilon)\rho_q(\boldsymbol{r}_1)|\Phi_A^0\Big\rangle \tag{1}$$

$G(z) = (z - H_A)^{-1}$ is the Green's function of the system. $\boldsymbol{q}, \omega$ are the momentum and energy transfer in an inclusive reaction from which S is extracted and $\phi(q,y)$ is the reduced response with the energy loss ω replaced by $y = (m/q)(\omega - q^2/2m)$ [4,5]. Benhar et al proposed [1,3]

$$\phi_B(q,y) = (q/m)S_B(q,\omega) = \mathrm{Re}\int dy'\phi_{PWIA}(q, y-y')R_B(q,y') \tag{2a}$$

$$\tilde{R}_B(q,s) = \int dy e^{-iys} R_B(q,y) \; = \; \exp\Big[\int d\boldsymbol{r}\rho_2(\boldsymbol{r},0;\boldsymbol{r},0)\exp\Big(i\tilde{\chi}_1(q,\boldsymbol{r};s)\Big) - 1\Big] \tag{2b}$$

$$\rightarrow \exp\Big[i\int d\boldsymbol{r}\rho_2(\boldsymbol{r},0;\boldsymbol{r},0)\tilde{\chi}_1(q,\boldsymbol{r};s)\Big] \tag{2c}$$

Here

$$(m/q)\phi_{PWIA} = S_{PWIA}(q,\omega) = \int d\boldsymbol{p}P\Big(\boldsymbol{p},\omega - \frac{(\boldsymbol{p}+\boldsymbol{q})^2}{2m}\Big), \tag{3}$$

is the response without core recoil in the PWIA in terms of the single particle spectral function P. R_B accounts for FSI and is expressed by means of the diagonal two-particle density matrix and an eikonal phase $\tilde{\chi}_1(q,\boldsymbol{r},s) = (m/q)\int_0^s ds' V(\boldsymbol{r} - s\hat{\boldsymbol{q}})$. It is totally off-shell in coordinate space with finite integration limits instead of $-\infty,\infty$, and reflects a particle, knocked-out inside the medium which will ultimately not be detected. Finally Eq. (2c) is the Born approximation of (2b) for weak V.

We now attempt to derive (2) within a framework, suitable for the discussion of approximations. It suffices to consider a non-relativistic, infinite system with Hamiltonian, states and energies $H_A = T + V, \Phi_A^n$ and E_A^n. Assuming local forces

$$\rho_q^\dagger(\boldsymbol{r}_1)H(\boldsymbol{p}_n,\boldsymbol{r}_n)\rho_q(\boldsymbol{r}_1) = H(\boldsymbol{p}_n,\boldsymbol{r}_n) + \mathcal{L}(\boldsymbol{p}_1,\boldsymbol{q}) \tag{4a}$$

$$\mathcal{L}(\boldsymbol{p}_1,\boldsymbol{q}) = (\boldsymbol{p}_1+\boldsymbol{q})^2/2m - \boldsymbol{p}_1^2/2m = \boldsymbol{q}^2/2m + p_{1z}q/m, \tag{4b}$$

where $\hat{\boldsymbol{q}} = \hat{\boldsymbol{z}}$. With a correspondingly shifted Green's function $\Gamma(z) = \rho_q^\dagger(\boldsymbol{r}_1)G(z)\rho_q(\boldsymbol{r}_1)$ (1) becomes [4]

$$S(q,\omega) = -(1/\pi)\mathrm{Im}\Big\langle \Phi_A^0|\Gamma(\omega + E_A^0 + i\epsilon)|\Phi_A^0\Big\rangle \tag{5a}$$

$$\Gamma = \Big(\omega + E_A^0 - H_A(\boldsymbol{r}_1;\boldsymbol{r}_j) + \mathcal{L}(\boldsymbol{p}_1,\boldsymbol{q}) + i\epsilon\Big)^{-1} \tag{5b}$$

$$= \Big(\omega + E_A^0 - H_{A-1}(\boldsymbol{r}_j) + T_{\boldsymbol{p}_1} + V_1(\boldsymbol{r}_1,\boldsymbol{r}_j) + \mathcal{L}(\boldsymbol{p}_1,\boldsymbol{q}) + i\epsilon\Big)^{-1}, \tag{5c}$$

with $V_1(\boldsymbol{r}_1;\boldsymbol{r}_j) = \Sigma_{j\leq 2}V(\boldsymbol{r}_1 - \boldsymbol{r}_j)$, the residual interaction of $'1'$ with the core.

In practice one computes the structure function S, retaining parts in the argument of Γ and disregarding the remainder γ. The corresponding response is then

$$S = S_d + \Delta S(\gamma_d) \approx S_d = S_{d,0} + S_{d,FSI}, \tag{6}$$

with some leading contribution $S_{d,0}$ and a Final State Interaction (FSA) part $S_{d,FSI}$. We discuss two choices

$$\Gamma_a = \Big(\omega + \mathcal{L}(\boldsymbol{p}_1, \boldsymbol{q}) - V_1(\boldsymbol{r}_1; \boldsymbol{r}_j) + i\epsilon\Big)^{-1}, \quad \gamma_a = E_A^0 - H_A + V_1(\boldsymbol{r}_1; \boldsymbol{r}_j) = E_A^0 - H_{A-1} - T_{\boldsymbol{p}_1} \tag{7a}$$

$$\Gamma_b = \Big(\omega + E_A^0 - H_{A-1}(j) + T_{\boldsymbol{p}_1+\boldsymbol{q}} + i\epsilon\Big)^{-1}, \quad \gamma_b = V_1(\boldsymbol{r}_1; \boldsymbol{r}_j), \tag{7b}$$

In a) the kinetic energy in H_A resides in the neglected part γ_a and thus suits a large-q approximation, where after absorbing $\boldsymbol{q}$, the recoiling particle $'1'$ has momentum $|\boldsymbol{p}_1 + \boldsymbol{q}| \approx q \gg p_1, p_j$. Its kinetic energy in the appropriate eikonal approximation is $\mathcal{L}$, Eq. (4b). For the reduced response one has for weak V [4]

$$\phi_a(q,y) = \frac{1}{2\pi}\mathrm{Re}\int_{-\infty}^{\infty} ds e^{isy}\Big[\frac{\rho_1(0,s)}{\rho} + i\int d\boldsymbol{r}\frac{\rho_2(\boldsymbol{r},0;\boldsymbol{r}-s\hat{\boldsymbol{q}},0)}{\rho}\tilde{\chi}(q,\boldsymbol{r};s) + ...\Big] \tag{8a}$$

$$\tilde{\chi} = \tilde{\chi}_1 + \tilde{\chi}_2 = (m/q)\Big[\int_0^s ds' V(\boldsymbol{r} - s'\hat{\boldsymbol{q}}) - sV(\boldsymbol{r} - s\hat{\boldsymbol{q}})\Big] \tag{8b}$$

Eqs. (8) give the lowest order terms in a $1/q$ expansion of ϕ in terms of non-diagonal density matrices ρ_1, ρ_2. In particular the asymptotic limit ($\boldsymbol{s} = s\hat{\boldsymbol{q}}$)

$$F_0(y) = \lim_{q\to\infty}\phi(q,y) = (4\pi)^{-2}\int_{|y|}^{\infty} dppn(p) = (4\pi)^{-2}\int_{|y|}^{\infty} dpp\int d\boldsymbol{s}\,\exp[i\boldsymbol{sp}](\rho_1(0,s)/\rho) \tag{9}$$

The first cumulant corresponding to (8) is [7]

$$\phi_a(q,y) = \frac{1}{2\pi}\mathrm{Re}\int dse^{isy}\frac{\rho_1(0,s)}{\rho}\tilde{R}_a(q,s) = \mathrm{Re}\int dy' F_0(y-y')R_a(q,y') \tag{10a}$$

$$\tilde{R}_a(q,s) = \exp\Big[\int d\boldsymbol{r}\frac{\rho_2(\boldsymbol{r},0;\boldsymbol{r}-s\hat{\boldsymbol{q}},0)}{\rho_1(0,s)}\Big(e^{i\tilde{\chi}(q,\boldsymbol{r};s)} - 1\Big)\Big] \to \exp\Big[i\int d\boldsymbol{r}\frac{\rho_2(\boldsymbol{r},0;\boldsymbol{r}-s\hat{\boldsymbol{q}},0)}{\rho_1(0,s)}\tilde{\chi}(q,\boldsymbol{r};s)\Big] \tag{10b}$$

In case b) one starts from the exact shifted Green's function except for the residual interaction V_1. Insertion of a complete set Φ_{A-1}^n into (5a) leads to the PWIA

$$S_b(q,\omega) = S_{PWIA}(q,\omega) = \int d\boldsymbol{p}P\Big(\boldsymbol{p}, \omega - \frac{(\boldsymbol{p}+\boldsymbol{q})^2}{2m}\Big), \tag{11}$$

with $P(p,E)$ the single-particle spectral function of the target.

According to (6), the above is the dominant part with no FSI left, unless one considers the otherwise neglected $\gamma_b = V_1(\boldsymbol{r}_1; \boldsymbol{r}_j)$. Inclusion of generally, non-diagonal core matrix-elements of V_1 causes grave complications. It has been suggested to replace $V_1(\boldsymbol{r}_1; \boldsymbol{r}_j)$ by an optical potential $\mathcal{V}^{opt}$

$$\langle \boldsymbol{p}, n|\gamma_b|\boldsymbol{p}', n'\rangle \rightarrow \langle \boldsymbol{p}, 0|V_1(i;j)|\boldsymbol{p}', 0\rangle \rightarrow \langle \boldsymbol{p}|\mathcal{V}^{opt}|\boldsymbol{p}'\rangle \tag{12}$$

However, by definition $\mathcal{V}^{opt}$ replaces only *diagonal* matrix elements of $V_1(\boldsymbol{r}_1; \boldsymbol{r}_j)$ and the approximation (12) is consequently impermissible.

With a fast recoiling particle also in b) one is tempted to introduce a Fixed Scatterers Approximation (FSA) for V_1. This, however, seems not commensurate with the retention of a dynamically active H_{A-1}. The objection is circumvented if $E_A^0 - H_{A-1}(j)$ in (5b) is replaced by an average separation energy $\langle\Delta\rangle$, or equivalently, if closure is applied to the states Φ_n^{A-1}, implicit in P, Eq. (11). Neglecting in addition $\boldsymbol{p}_1^2 \ll \mathcal{L}$ one finds

$$\phi_b^{clos}(q,y) \rightarrow \phi_a(q, y - \langle\Delta\rangle) \tag{13}$$

Except for a small shift, relevant only around the quasi-elastic peak $y = 0$, closure reduces case b) to a).

We return to the expression (2) of Benhar *et al.* [1,3], passing over many intermediate heuristic steps in its construction. Its form resembles (10) which without proof has been assumed to also hold if the leading asymptotic part $S_{as}(= F_0) \rightarrow S_{PWIA}$. The discussion of case b) shows this not to be possible, even when assuming (12).

Next the authors claim (2) to be the same as used by Silver [9], itself a re-derivation of results in [4,7], but there are obvious differences. We mention the diagonal ρ_2 in (2), different from the semi-diagonal in (10b) and also one potential term out of the two in (8b). The second one vanishes only for hard-core interactions, using a non-diagonal ρ_2.

Putting aside the derivation of (2) we next assess the actual difference, comparing $1/q$ expansions of ϕ_a, ϕ_{PWIA}, which can be shown to have the same asymptotic limit $F_0(y)$, Eq. (9). From Eqs. (8) and (2) one finds for the Fourier transforms of the lowest order FSI term $\phi_1(q,y) = (m/q)F_1(y)$

$$\tilde{F}_{a,1}(s) = +i\int d\boldsymbol{r}\frac{\rho_2(\boldsymbol{r},0;\boldsymbol{r}-s\hat{\boldsymbol{q}},0)}{\rho}\Big[\int_0^s ds' V(\boldsymbol{r}-s'\hat{\boldsymbol{q}}) - sV(\boldsymbol{r}-s\hat{\boldsymbol{q}})\Big] \tag{14a}$$

$$\begin{aligned}\tilde{F}_{PWI,1}(s) &= \Sigma_n \int d\boldsymbol{p}\, e^{ip_z s}\, |\Gamma_n(p)|^2\Big(\frac{p^2}{2m} - \Delta_n\Big) + i\int d\boldsymbol{r}\frac{\rho_2(\boldsymbol{r},0;\boldsymbol{r},0)}{\rho}\int_0^s ds' V(\boldsymbol{r}-s'\hat{\boldsymbol{q}}) \\ &= i\int d\boldsymbol{r}\Big[\frac{\rho_2(\boldsymbol{r},0;\boldsymbol{r},0)}{\rho}\int_0^s ds' V(\boldsymbol{r}-s'\hat{\boldsymbol{q}}) - \frac{\rho_2(\boldsymbol{r},0;\boldsymbol{r}-s\hat{\boldsymbol{q}},0)}{\rho} sV(\boldsymbol{r}-s\hat{\boldsymbol{q}})\Big] \qquad (14b)\end{aligned}$$

The proof leading to the second equation in (14b) can be found in [10]. One notices diagonal and semi-diagonal ρ_2 as opposed to exclusively the half-diagonal one in (14a). The two expressions (14) thus differ and so do their first cumulants. Numerical consequences have been discussed in a comparison of inclusive cross section of electrons from Fe and Nuclear Matter [2]. In particular for low energy losses there are considerable differences in the relative magnitude of the leading and FSI parts, as well as in the total results for (10).

We return to the simplifying assumptions made in the introduction: Should short-range repulsion in V produce large integrals in (10c), their contribution may be tempered by a re-summation $V \to V_{eff} = t$ [11] (cf. Eq. (2b)). Also, the above presentation is not different for finite targets, nor is it qualitatively modified when minimal relativistic effects [1,2] are applied.

We conclude that, starting from a well-defined theory, there seems to be no way to actually derive the total response in a form (2) which modifies a leading PWIA by FSI terms.

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